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Journal of Geometry and Physics 48 (2003) 135–147

JOURNAL OF GEOMETRY AND PHYSICS

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# Partition functions and Jacobi fields in the Morse theory

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Received 7 January 2003

**Abstract**

We study the semiclassical partition function in the frame work of the Morse theory, to clarify the phase factor of the partition function and to relate it to the eta invariant of Atiyah. Converting physical system with potential into a curved manifold, we exploit the Jacobi fields and their corresponding eigenvalues of the Sturm–Liouville operator to be associated with geodesics on the curved manifold and with the Hamilton–Jacobi theory.


*MSC:* 81T99

*Subj. Class.:* Quantum field theory

*Keywords:* Morse theory; Jacobi field; Partition function; Eta invariant

## 1. Introduction

Nowadays there have been considerable discussions concerning the topological invariants such as the Euler characteristics, the Hopf invariant in mathematical physics. Moreover, the eta invariant of Atiyah et al. [1,2] has been studied in quantum field theory associated with the Jones polynomial and knot theory [2] and even in hadron physics such as the chiral bag model [3]. Recently, the Yamabe invariant [4,5] is also investigated in general relativity associated with the topology of boundary surface of black holes with nontrivial higher genus.

On the other hand, since Feynman [6] proposed the path integral formalism, there have been tremendous developments in quantum field theory, and the partition functions in the

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doi:10.1016/S0393-0440(03)00028-7

path integral formalism have become crucial in investigating many aspects of recent theoretical physics. Moreover, the supersymmetric quantum mechanics has been exploited by Witten [7] to discuss the Morse inequalities [8–10]. The Morse indices for pair of critical points of the symplectic action function have been also investigated based on the spectral flow of the Hessian of the symplectic function [11], and on the Hilbert spaces the Morse homology has been considered to discuss the critical points associated with the Morse index [12]. Recently, the semiclassical partition function has been derived in the Chern–Simons gauge theory exploiting the invariant integration scheme [13]. Even though Morette studied the partition functions semiclassically long ago, her expression for its phase factor still remains unclear to possess somehow subtleties [14].

In this paper we reformulate the semiclassical partition function in the frame work of the Morse theory [8–10], to clarify the phase factor of the partition function and to relate it to the eta invariant of Atiyah et al. [1,2]. To do this, we will convert the physical system with potential into a curved manifold, on which we will use the Jacobi fields and their eigenvalues of the Sturm–Liouville operator associated with the geodesics on the curved manifold and with the Hamilton–Jacobi theory.

In Section 2, we will consider the Morse theory on the manifold constructed via the potential of a conservative physical system, to discuss the eta invariant of Atiyah involved in the semiclassical partition function of the system. In Section 3, we will consider the Jacobi equation to describe the semiclassical partition function in terms of the Van Vleck determinant [15] by introducing a smooth one-parameter family of geodesics on the manifold.

## 2. Morse theory and eta invariant

In this section we consider a particle in a conservative physical system with constant energy to relate the eta invariant with the Morse theory.

**Proposition 2.1.** *Let $E = T + V$ be the constant total energy for a particle of mass $m$ in a conservative physical system on a flat manifold with the Minkowski metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$, where $T$ and $V$ are the kinetic and potential energies, respectively, then we have a curved manifold $M$ defined as the four-metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$ with*

$$g_{ab} = \frac{m(E-2V)^2}{2(E-V)}\delta_{ab}, \tag{2.1}$$

*and on which the action of the particle is given by*

$$S = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau (g_{ab}v^a v^b)^{1/2} \tag{2.2}$$

*with the proper time $\tau$ ($\tau_1 \le \tau \le \tau_2$) and the vector field $v^a = (\partial/\partial\tau)^a$.*

**Proof.** Define the kinetic energy $T$ and potential energy $V$ as

$$T = \frac{1}{2} m \delta_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}, \qquad V = V(x^i).$$

Since $E = T + V$, we can rewrite the Lagrangian $L = T - V$ as follows:

$$L = \left[ \frac{m(E-2V)^2}{2(E-V)} \delta_{ij} \frac{\mathrm{d}x^i}{\mathrm{d}\tau} \frac{\mathrm{d}x^j}{\mathrm{d}\tau} \right]^{1/2} = (g_{ij} v^i v^j)^{1/2},$$

where $v^i = \mathrm{d}x^i/\mathrm{d}\tau$ and

$$g_{ij} = \frac{m(E-2V)^2}{2(E-V)} \delta_{ij}.$$

The action is then given by

$$S = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, L = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau (g_{ij} v^i v^j)^{1/2}.$$

On the curved manifold $M$ with the above metric $g_{ij}$ originated from the potential energy $V(x^i)$, in the abstract index notation we arrive at the metric (2.1) and the action (2.2). □

Here one notes that, without loss of generality, $V(x^i)$ can be chosen to vanish at starting point at $\tau_1$ and the metric $g_{ab}$ in (2.1) does not have any singularities since its denominator is positive definite. With the metric $g_{ab}$ in mind, one can define a unique covariant derivative $\nabla_a$ satisfying $\nabla_a g_{bc} = 0$.

**Proposition 2.2.** *In the stationary phase approximation where the absolute value of the deviation vector $|w^a(\tau)|$ is infinitesimally small, for a particle in a conservative physical system, one can expand the action $S$ around the geodesic $C_0$*

$$S = S_{\mathrm{cl}} + S^{(1)}(w^a) + \tfrac{1}{2} S^{(2)}(w^a) + \cdots, \tag{2.3}$$

*where $S_{\mathrm{cl}}$ is a classical action and the next order terms are given by*

$$S^{(1)}(w^a) = -\int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, w^b v^a \nabla_a v_b, \tag{2.4}$$

$$S^{(2)}(w^a) = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, g_{ab} w^a \Lambda^b_c w^c \tag{2.5}$$

*with the Sturm–Liouville operator given by*

$$\Lambda^a_b = -\delta^a_b \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - R^a_{cbd} v^c v^d. \tag{2.6}$$

**Proof.** Consider a smooth one-parameter family of curves $C_\alpha(\tau)$, parameterized by a proper time $\tau$ ($\tau_1 \le \tau \le \tau_2$) such that for all $\alpha$ and $p, q \in M$, $C_\alpha(\tau_1) = p$, $C_\alpha(\tau_2) = q$, and $C_0$ is a geodesic, along which a tangent vector field $v^a$ satisfies the geodesic equation

$$v^a \nabla_a v^i = \frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}\tau} \frac{\mathrm{d}x^k}{\mathrm{d}\tau} = 0, \tag{2.7}$$

where $\nabla_a v^b = \partial_a v^b + \Gamma^b_{ac} v^c$. Let $\Sigma$ be a two-dimensional submanifold spanned by curves $C_\alpha(\tau)$ and we choose $(\tau, \alpha)$ as coordinates of $\Sigma$. The vector fields $v^a = (\partial/\partial\tau)^a$ and $w^a =$

$(\partial/\partial\alpha)^a$ are then the tangent to the family of curves and the deviation vector representing the displacement to an infinitesimally nearby curve, respectively. Without loss of generality, $w^a$ can be chosen orthogonal to $v^a$ and vanishes at end-points to yield the boundary conditions,

$$w^a(\tau_1) = w^a(\tau_2) = 0. \tag{2.8}$$

Since $v^a$ and $w^a$ are coordinate vector fields, they commute to each other,

$$\pounds_v w^a = v^b\nabla_b w^a - w^b\nabla_b v^a = 0. \tag{2.9}$$

In the stationary phase approximation where $|w^a|$ is infinitesimally small, one can expand the action (2.2) around the geodesic $C_0$ as in (2.3). For simplicity, we parameterize the curve so that the Lagrangian can be given by $L = (g_{ab}v^a v^b)^{1/2} = 1$ along the geodesic without loss of generality, since the action (2.2) is parameterization independent. The classical action is then given by $S_{\rm cl} = S|_{\alpha=0}$ and the next order terms are given as follows:

$$S^{(1)}(w^a) = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, w^a\nabla_a(v^b v_b)^{1/2} = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, v_b w^a \nabla_a v^b = -\int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, w^b v^a \nabla_a v_b,$$

$$\begin{aligned}
S^{(2)}(w^a) &= -\int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, w^c\nabla_c(w_b v^d\nabla_d v^b)\\
&= -\int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, w^c w_b(\nabla_c v^d\nabla_d v^b + v^d\nabla_c\nabla_d v^b)\\
&= -\int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, w_b(v^c\nabla_c w^d\nabla_d v^b + w^c v^d\nabla_d\nabla_c v^b - R^b_{cde}w^c v^d v^e)\\
&= -\int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, g_{ab}w^a(v^c\nabla_c(v^d\nabla_d w^b) + R^b_{cde}v^c v^e w^d) = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, g_{ab}w^a\Lambda^b_c w^c
\end{aligned}$$

with the Sturm–Liouville operator $\Lambda^a_b = -\delta^a_b(\mathrm{d}^2/\mathrm{d}\tau^2) - R^a_{cbd}v^c v^d$. Here we have put the condition $\alpha = 0$ at the end of the calculations of $S^{(1)}(w^a)$ and $S^{(2)}(w^a)$ and we have used (2.7)–(2.9) and the convention for the Riemann curvature tensor for any vector field $v^a$ [10]

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)v^c = -R^c_{abd}v^d. \qquad \square \tag{2.10}$$

**Proposition 2.3.** *In the stationary phase approximation where the absolute value of the deviation vector $|w^a(\tau)|$ is infinitesimally small, for a particle in a conservative physical system, one can have a partition function of the form*

$$Z(\tau_2,\tau_1) = \mathrm{e}^{\mathrm{i}S_{\rm cl}}\int D[w^a(\tau)]\,\mathrm{e}^{\mathrm{i}\sum_{m,n}c_{mn}a^m a^n}, \tag{2.11}$$

*where the deviation vector $w^a(\tau)$ is given in terms of superposition of an orthonormal basis $\{u^a_n(\tau)\}$ in the Hilbert space,*

$$w^a(\tau) = \sum_{n=1}^{\infty} a^n u^a_n(\tau), \tag{2.12}$$

*and*

$$c_{mn} = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, g_{ab} u_m^a \Lambda_c^b u_n^c = (u_m, \Lambda u_n). \tag{2.13}$$

**Proof.** Consider a partition function [16]

$$Z(\tau_2, \tau_1) = \int D[x^i(\tau)]\, \mathrm{e}^{\mathrm{i}S(x^i)}, \tag{2.14}$$

which, in the stationary phase approximation, contains a widely oscillatory integral [2,17,18] and is thus given by contributions from the points of stationary phase. Here the stationary points precisely construct the geodesic, along which the total energy is constant. Since the above $S^{(1)}(w^a)$ in (2.4) vanishes due to the geodesic equation (2.7), by inserting the action $S$ in (2.3) into $Z$ in (2.14), we obtain the partition function of the form

$$Z(\tau_2, \tau_1) = \mathrm{e}^{\mathrm{i}S_{\mathrm{cl}}} \int D[w^a(\tau)]\, \mathrm{e}^{\mathrm{i}S^{(2)}(w^a)/2}. \tag{2.15}$$

Define a scalar product

$$(w, w') = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, g_{ab} w^a(\tau) w'^b(\tau) \tag{2.16}$$

to transform the path space of the integral (2.15) into the Hilbert space which is an external product of two spaces: three-dimensional space of the physical system and the infinite-dimensional Hilbert space of the continuous scalar functions on $[\tau_1, \tau_2]$ vanishing at $\tau_1$ and $\tau_2$. By choosing an orthonormal basis $\{u_n^a(\tau)\}$ in this Hilbert space, we have the deviation vector $w^a(\tau)$ in terms of superposition of $\{u_n^a(\tau)\}$ as in (2.12) so that, together with the scalar product (2.16), we can rewrite the action (2.5) as

$$S^{(2)}(w^a) = \sum_{m,n} c_{mn} a^m a^n \tag{2.17}$$

with the $c_{mn}$ in (2.13). Inserting (2.17) into (2.15) we arrive at the partition function (2.11). □

**Proposition 2.4.** *In the stationary phase approximation where the absolute value of the deviation vector $|w^a(\tau)|$ is infinitesimally small, the partition function for a particle in a conservative physical system can be rewritten as*

$$Z(\tau_2, \tau_1) = J\, \mathrm{e}^{\mathrm{i}S_{\mathrm{cl}}}\, \mathrm{e}^{\mathrm{i}\pi \sum_n \operatorname{sign} \lambda_n/4} \prod_{n=1}^{\infty} \left| \frac{2\pi}{\lambda_n} \right|^{1/2}, \tag{2.18}$$

*where $J$ is the Jacobian associated with the transformation from the deviation vector $w^a(\tau)$ to the orthonormal basis $\{u_n^a(\tau)\}$ in the Hilbert space, and $\lambda_n$ are the eigenvalues of the Sturm–Liouville operator $\Lambda_b^a = -\delta_b^a(\mathrm{d}^2/\mathrm{d}\tau^2) - R^a_{cbd} v^c v^d$,*

$$-\Lambda_b^a u_n^b + \lambda_n u_n^a = 0 \tag{2.19}$$

*with the boundary conditions of the eigenfunction*

$$u_n^a(\tau_1) = u_n^a(\tau_2) = 0. \tag{2.20}$$

*Note that the phase factor is proportional to* $\sum_n \operatorname{sign}\lambda_n$, *which is associated with the eta invariant of Atiyah et al.* [1,2]

$$\eta = \frac{1}{2}\lim_{s\to 0} \operatorname{sign}\lambda_n |\lambda_n|^{-s}. \tag{2.21}$$

**Proof.** In order to diagonalize the matrix $c_{mn}$ in (2.13), we find an orthonormal basis of eigenfunctions $\{u_n^a(\tau)\}$ of the Sturm–Liouville operator $\Lambda^a_b$ with eigenvalues $\lambda_n$ via the following eigenvalue equations in the Morse theory [8–10] in their component form:

$$-\Lambda^i_j u^j_n + \lambda_n u^i_n = \frac{\mathrm{d}^2 u^i_n}{\mathrm{d}\tau^2} + R^i_{kjl} v^k v^l u^j_n + \lambda_n u^i_n = 0$$

with the boundary conditions of the eigenfunction originated from (2.8)

$$u^i_n(\tau_1) = u^i_n(\tau_2) = 0, \quad i = 1, 2, 3.$$

With the above eigenvalues and eigenfunctions, we obtain

$$c_{mn} = \lambda_m (u_m, u_n) = \lambda_m \delta_{mn}$$

to yield

$$S^{(2)}(w^a) = \sum_n \lambda_n (a^n)^2$$

from which we rewrite the partition function (2.11) as follows:

$$Z(\tau_2, \tau_1) = J\, \mathrm{e}^{\mathrm{i}S_{\mathrm{cl}}} \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}a^n\, \mathrm{e}^{\mathrm{i}\lambda_n (a^n)^2/2}.$$

Here $J$ is the Jacobian defined as

$$D[w^a] = J \prod_{n=1}^{\infty} \mathrm{d}a^n,$$

and is independent of $w^a(\tau)$ due to the linearity of the transformation (2.12), so that $J$ can be brought out of the integral symbol.

By taking the vanishing $\epsilon$ limit of the absolutely convergent integral, one can obtain

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}a^n\, \mathrm{e}^{\mathrm{i}\lambda_n (a^n)^2/2}\, \mathrm{e}^{-\epsilon (a^n)^2} = \mathrm{e}^{\mathrm{i}\pi \operatorname{sign}\lambda_n/4} \left|\frac{2\pi}{\lambda_n}\right|^{1/2}$$

to arrive at the partition function (2.18). □

**Proposition 2.5.** *In the stationary phase approximation where the absolute value of the deviation vector* $|w^a(\tau)|$ *is infinitesimally small, the partition function for a particle in a conservative physical system can be rewritten as*

$$Z(\tau_2, \tau_1) = J\, \mathrm{e}^{\mathrm{i}(S_{\mathrm{cl}} - \hat{S}_{\mathrm{cl}})}\, \mathrm{e}^{\mathrm{i}\pi \sum_n (\operatorname{sign}\lambda_n - \operatorname{sign}\hat{\lambda}_n)/4} \left|\frac{\prod_n \hat{\lambda}_n}{\prod_n \lambda_n}\right|^{1/2} \hat{Z}(\tau_2, \tau_1), \tag{2.22}$$

*where $\hat{\lambda}_n$ are the eigenvalues of the Sturm–Liouville operator $\hat{\Lambda}^a_b = -\delta^a_b(\mathrm{d}^2/\mathrm{d}\tau^2)$*

$$-\hat{\Lambda}^a_b\hat{u}^b_n + \hat{\lambda}_n\hat{u}^a_n = 0 \tag{2.23}$$

*with the boundary conditions of the eigenfunction*

$$\hat{u}^a_n(\tau_1) = \hat{u}^a_n(\tau_2) = 0 \tag{2.24}$$

*for a free particle with the same constant energy $E$ as that of the particle in a conservative physical system, and $\hat{Z}(\tau_2, \tau_1)$ is the corresponding partition function.*

**Proof.** We relate the geodesic of the particle with constant energy $E$ in a conservative physical system to that of a free particle with the same energy $E$. Recalling that the Jacobian $J$ remains unchanged [16,19] for the unitary transformation $\{u^a_n(\tau)\} \to \{\hat{u}^a_n(\tau)\}$, in the limit $V \to 0$, (2.1) yields the free particle metric

$$\hat{g}_{ij} = \tfrac{1}{2}mE\delta_{ij}, \tag{2.25}$$

and (2.18) produces the partition function

$$\hat{Z}(\tau_2, \tau_1) = J\,\mathrm{e}^{\mathrm{i}\hat{S}_{\mathrm{cl}}}\,\mathrm{e}^{\mathrm{i}\pi\sum_n \mathrm{sign}\,\hat{\lambda}_n/4}\prod_{n=1}^{\infty}\left|\frac{2\pi}{\hat{\lambda}_n}\right|^{1/2}, \tag{2.26}$$

and the eigenvalue equations of the Sturm–Liouville operator $\hat{\Lambda}^a_b = -\delta^a_b(\mathrm{d}^2/\mathrm{d}\tau^2)$ can be described in terms of their component form

$$-\hat{\Lambda}^i_j\hat{u}^j_n + \hat{\lambda}_n\hat{u}^i_n = \frac{\mathrm{d}^2\hat{u}^i_n}{\mathrm{d}\tau^2} + \hat{\lambda}_n\hat{u}^i_n = 0$$

with

$$\hat{u}^i_n(\tau_1) = \hat{u}^i_n(\tau_2) = 0, \quad i = 1, 2, 3.$$

Combination of (2.18) and (2.26) yields the partition function (2.22). □

## 3. Jacobi fields and partition functions

Now we consider the Jacobi equation to express the absolute value of the ratio in (2.22) in terms of the initial data at the starting point $p = x^i(\tau_1)$, by introducing a smooth one-parameter family of geodesics $\gamma_n(\tau)$ on the manifold $M$. Here one can vary the parameter $\alpha \in R$ by infinitesimally changing the direction of the initial velocity $v^i(\tau_1) = (\mathrm{d}x^i/\mathrm{d}\tau)(\tau_1)$ at $p$, and also one can choose $\alpha$ and $\tau$ as coordinates of a submanifold $\Sigma_\gamma$ spanned by the geodesics $\gamma_\alpha(\tau)$ on $M$. Along the geodesic $\gamma_0$, one can have tangent vector fields $v^a$ and deviation vector field $w^a$ which points to an infinitesimally nearby geodesic and vanishes at $p$, namely, $w^a(\tau_1) = 0$, so that one can have the relative acceleration of the displacement to an infinitesimally nearby geodesic

$$a^a = v^c\nabla_c(v^b\nabla_b w^a) = -R^a_{bcd}v^b v^d w^c. \tag{3.1}$$

Here note that, differently from (2.4) and (2.5), one can exploit the fact that all the curves involved in $a^a$ are geodesics, so that one can use the identity $w^c\nabla_c(v^b\nabla_b v^a) = 0$. The relative acceleration (3.1) then yields the geodesic deviation equation

$$v^c\nabla_c(v^b\nabla_b w^a) + R^a_{bcd}v^b v^d w^c = 0, \tag{3.2}$$

or its component form

$$\frac{\mathrm{d}^2 w^i}{\mathrm{d}\tau^2} + R^i_{kjl}v^k v^l w^j = -\Lambda^i_j w^j = 0, \tag{3.3}$$

which is also known as a Jacobi equation. Note that the solution $w^a$ of (3.2) is named a Jacobi field on the geodesic $\gamma_0$ whose tangent is $v^a$.

**Proposition 3.1.** *In the stationary phase approximation where the absolute value of the deviation vector $|w^a(\tau)|$ is infinitesimally small, the partition function for a particle in a conservative physical system can be rewritten as*

$$Z(\tau_2, \tau_1) = (2\pi \mathrm{i})^{-3/2}\,\mathrm{e}^{\mathrm{i}S_{\mathrm{cl}}}\,\mathrm{e}^{\mathrm{i}\pi\sum_n(\mathrm{sign}\,\lambda_n - \mathrm{sign}\,\hat{\lambda}_n)/4}\left|\det \hat{g}_{ij}(\tau_1)\frac{\partial v^j(\tau_1)}{\partial x^k(\tau_2)}\right|^{1/2}, \tag{3.4}$$

*where $\hat{\lambda}_n$ are the eigenvalues of the Sturm–Liouville operator $\hat{\Lambda}^a_b = -\delta^a_b(\mathrm{d}^2/\mathrm{d}\tau^2)$ with the boundary conditions of the eigenfunction, $\hat{u}^a_n(\tau_1) = \hat{u}^a_n(\tau_2) = 0$ for a free particle with the same constant energy $E$ as that of the particle in the conservative physical system.*

**Proof.** Since the Jacobi equation (3.3) is a linear differential equation, the Jacobi field $w^i(\tau)$ depends linearly on the inertial data $w^i(\tau_1)$ and $(\mathrm{d}w^i/\mathrm{d}\tau)(\tau_1)$ at the starting point $p$ to yield

$$w^i(\tau) = T^i_j(\tau)\frac{\mathrm{d}w^j}{\mathrm{d}\tau}(\tau_1), \tag{3.5}$$

where

$$T^i_j(\tau_1) = 0, \qquad \frac{\mathrm{d}T^i_j}{\mathrm{d}\tau}(\tau_1) = \delta^i_j, \tag{3.6}$$

and $T^i_j(\tau)$ can be defined as

$$T^i_j(\tau) = \frac{\mathrm{d}x^i(\tau)}{\mathrm{d}v^j(\tau_1)}. \tag{3.7}$$

Here note that, since the coordinates $x^i(\tau)$ and the velocity $v^i(\tau) = (\mathrm{d}x^i/\mathrm{d}\tau)(\tau)$ are independent variables at the same time, say $\tau_1$, one can easily check that the definition (3.7) satisfies (3.6). Substituting (3.5) into (3.3), one can rewrite the Jacobi equation in terms of $T^i_j$ as follows:

$$\frac{\mathrm{d}^2 T^i_j}{\mathrm{d}\tau^2} + R^i_{klh}v^k v^h T^l_j = -\Lambda^i_k T^k_j = 0.$$

Similarly, for the free particle one can obtain

$$\frac{\mathrm{d}^2\hat{T}^i_j}{\mathrm{d}\tau^2} = -\hat{\Lambda}^i_l\hat{T}^l_j = 0, \tag{3.8}$$

$$\hat{T}^i_j(\tau_1) = 0, \qquad \frac{\mathrm{d}\hat{T}^i_j}{\mathrm{d}\tau}(\tau_1) = \delta^i_j \tag{3.9}$$

from which the ratio in (2.22) can be rewritten as [20]

$$\left|\frac{\prod_n \hat{\lambda}_n}{\prod_n \lambda_n}\right| = \left|\frac{\det \hat{T}^i_j(\tau_2)}{\det T^i_j(\tau_2)}\right|. \tag{3.10}$$

On the other hand, in order to evaluate explicitly the partition function for the free particle, we introduce the expression for $D[w^a]$ [16,21] associated with the metric (2.25)

$$D[w^a(\tau)] = \prod_{n=0}^{N-1}\left[\det\left(\frac{\hat{g}_{ab}}{2\pi\mathrm{i}\Delta\tau}\right)\right]^{1/2}\prod_{n=0}^{N-1}\mathrm{d}w^a_n \tag{3.11}$$

with $\Delta\tau = (\tau_2-\tau_1)/N$, $w^a_n = w^a(\tau_1+n\Delta\tau)$ and the boundary conditions

$$w^a_0 = w^a(\tau_1) = 0, \qquad w^a_N = w^a(\tau_2) = 0.$$

Moreover, $\hat{S}^{(2)}(w^a)$ can then be rewritten as

$$\begin{aligned}\hat{S}^{(2)}(w^a) &= \int_{\tau_1}^{\tau_2}\mathrm{d}\tau\,\frac{\mathrm{d}w^a}{\mathrm{d}\tau}\hat{g}_{ab}\frac{\mathrm{d}w^b}{\mathrm{d}\tau} = \frac{1}{\Delta\tau}\sum_{n=0}^{N-1}(w^a_{n+1}-w^a_n)\hat{g}_{ab}(w^b_{n+1}-w^b_n)\\ &= \frac{1}{\Delta\tau}\sum_{n=1}^{N-1}(2w^a_n\hat{g}_{ab}w^b_n - w^a_n\hat{g}_{ab}w^b_{n+1} - w^a_{n+1}\hat{g}_{ab}w^b_n)\\ &= \frac{1}{\Delta\tau}\sum_{n=0}^{N-1}(w^a_n - w^c_{n+1}h^{ca}_n)G^{ab}(w^b_n - h^{bd}_n w^d_{n+1}),\end{aligned} \tag{3.12}$$

where the matrices $G_n$ and $h_n$ satisfy the following relations:

$$\begin{aligned}&G_1 = 2\hat{g}, \qquad G_{n+1} + h_nG_nh_n = 2\hat{g}, \quad n = 1,\ldots,N-2,\\ &G_nh_n = h_nG_n = \hat{g}, \quad n = 1,\ldots,N-1,\end{aligned} \tag{3.13}$$

which yield

$$\begin{aligned}&h_n = G_n^{-1}\hat{g}, \quad n = 1,\ldots,N-1,\\ &G_{n+1} + \hat{g}G_n^{-1}\hat{g} - 2\hat{g} = 0,\\ &n = 1,\ldots,N-2.\end{aligned} \tag{3.14}$$

Now, consider a linear transformation,

$$z^a_n = w^a_n - h^{ab}_n w^b_{n+1}, \quad n = 1,\ldots,N-1, \tag{3.15}$$

whose Jacobian is trivially given by $J(w^a \to z^a) = 1$. Exploiting (3.11), (3.12) and (3.15), we can evaluate the partition function for the free particle as follows:

$$\begin{aligned}
\hat{Z}(\tau_2, \tau_1) &= \int \prod_{n=0}^{N-1} \left[\det\left(\frac{\hat{g}}{2\pi \mathrm{i} \Delta\tau}\right)\right]^{1/2} \prod_{n=1}^{N-1} \mathrm{d}z_n \, \mathrm{e}^{\mathrm{i}(\hat{S}_{\mathrm{cl}} + (1/(2\Delta\tau)) \sum_{n=1}^{N-1} z_n G_n z_n)} \\
&= \mathrm{e}^{\mathrm{i}\hat{S}_{\mathrm{cl}}} \prod_{n=0}^{N-1} \left[\det\left(\frac{\hat{g}}{2\pi \mathrm{i} \Delta\tau}\right)\right]^{1/2} \prod_{n=1}^{N-1} \int \mathrm{d}z_n \, \mathrm{e}^{-(1/(2\mathrm{i}\Delta\tau)) z_n G_n z_n} \\
&= \mathrm{e}^{\mathrm{i}\hat{S}_{\mathrm{cl}}} \prod_{n=0}^{N-1} \left[\det\left(\frac{\hat{g}}{2\pi \mathrm{i} \Delta\tau}\right)\right]^{1/2} \prod_{n=1}^{N-1} [\det(2\pi \mathrm{i} \Delta\tau) \det G_n^{-1}]^{1/2} \\
&= \mathrm{e}^{\mathrm{i}\hat{S}_{\mathrm{cl}}} \left[\det\left(\frac{\hat{g}}{2\pi \mathrm{i}}\right)\right]^{1/2} \left[\det\left(\Delta\tau \prod_{n=1}^{N-1} G_n \hat{g}^{-1}\right)^{-1}\right]^{1/2} \\
&= (2\pi \mathrm{i})^{-3/2} \, \mathrm{e}^{\mathrm{i}\hat{S}_{\mathrm{cl}}} \left(\frac{\det \hat{g}}{\det \hat{T}_{N-1}}\right)^{1/2},
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
&\hat{T}_n = \Delta\tau G_1 \hat{g}^{-1} G_2 \hat{g}^{-1} \cdots G_n \hat{g}^{-1}, \quad n = 1, \ldots, N-1, \\
&\hat{T}_{n+1} = \hat{T}_n G_{n+1} \hat{g}^{-1} = \hat{T}_{n-1} G_n \hat{g}^{-1} G_{n+1} \hat{g}^{-1},
\end{aligned} \tag{3.17}$$

from which we have the identity

$$(\hat{T}_{n+1} - \hat{T}_n) - (\hat{T}_n - \hat{T}_{n-1}) = 0. \tag{3.18}$$

In the limit of $N \to \infty$, (3.13), (3.14) and (3.17) yield the initial conditions

$$\hat{T}^i_j(\tau_1) = 0, \qquad \frac{\mathrm{d}\hat{T}^i_j}{\mathrm{d}\tau}(\tau_1) = \delta^i_j,$$

which are the same as (3.9). Moreover, in the limit of $N \to \infty$, (3.18) can be now written as the differential equation

$$\frac{\mathrm{d}^2 \hat{T}^i_j}{\mathrm{d}\tau^2} = 0,$$

equivalent to (3.8), and the partition function (3.16) for the free particle yields

$$\hat{Z}(\tau_2, \tau_1) = (2\pi \mathrm{i})^{-3/2} \, \mathrm{e}^{\mathrm{i}\hat{S}_{\mathrm{cl}}} \left(\frac{\det \hat{g}_{ij}(\tau_1)}{\det \hat{T}^i_j(\tau_2)}\right)^{1/2}, \tag{3.19}$$

where we have used the fact that $\hat{g}_{ij}$ is a constant independent of $\tau$.

Using the above results (3.10) and (3.19), the partition function (2.22) can be rewritten as

$$Z(\tau_2, \tau_1) = (2\pi \mathrm{i})^{-3/2} \, \mathrm{e}^{\mathrm{i}S_{\mathrm{cl}}} \, \mathrm{e}^{\mathrm{i}\pi \sum_n (\mathrm{sign}\,\lambda_n - \mathrm{sign}\,\hat{\lambda}_n)/4} \left|\frac{\det \hat{g}_{ij}(\tau_1)}{\det T^i_j(\tau_2)}\right|^{1/2} \tag{3.20}$$

to yield the partition function (3.4) for the particle in a conservative physical system. □

**Proposition 3.2.** *In the stationary phase approximation where the absolute value of the deviation vector $|w^a(\tau)|$ is infinitesimally small, the partition function for a particle in a conservative physical system can be rewritten as*

$$Z(\tau_2,\tau_1) = (2\pi \mathrm{i})^{-3/2}\, \mathrm{e}^{\mathrm{i}\pi\sum_n(\mathrm{sign}\,\lambda_n - \mathrm{sign}\,\hat{\lambda}_n)/4}\, \mathrm{e}^{\mathrm{i}S_{\mathrm{cl}}} \left|\det \frac{\partial^2 S_{\mathrm{cl}}}{\partial x^i(\tau_1)\partial x^k(\tau_2)}\right|^{1/2}, \tag{3.21}$$

*where $\hat{\lambda}_n$ are the eigenvalues of the Sturm–Liouville operator $\hat{\Lambda}^a_b = -\delta^a_b(\mathrm{d}^2/\mathrm{d}\tau^2)$ with the boundary conditions of the eigenfunction, $\hat{u}^a_n(\tau_1) = \hat{u}^a_n(\tau_2) = 0$, for a free particle with the same constant energy E as that of the particle in a conservative physical system. Note that the determinant involved here is known as the Van Vleck determinant* [15].

**Proof.** Consider the Hamilton–Jacobi theory [22] where classical conjugate momentum $p^{\mathrm{cl}}_i(\tau)$ corresponding to $x^i(\tau)$ in the action (2.2) is given by

$$p^{\mathrm{cl}}_i(\tau) = \frac{\partial L_{\mathrm{cl}}}{\partial v^i} = g_{ij}v^j, \qquad p^{\mathrm{cl}}_i(\tau_1) = \frac{\partial S_{\mathrm{cl}}}{\partial x^i(\tau_1)}$$

from which we obtain

$$\hat{g}_{ij}(\tau_1) = \frac{\partial p^{\mathrm{cl}}_i(\tau_1)}{\partial v^j(\tau_1)}, \qquad \frac{\partial v^i(\tau_1)}{\partial x^j(\tau_2)} = \frac{1}{\hat{g}_{ik}(\tau_1)}\frac{\partial p^{\mathrm{cl}}_k(\tau_1)}{\partial x^j(\tau_2)} = \frac{1}{\hat{g}_{ik}(\tau_1)}\frac{\partial^2 S_{\mathrm{cl}}}{\partial x^k(\tau_1)\partial x^j(\tau_2)}$$

to, together with (2.15) and (3.4), yield the desired semiclassical partition function (3.21). □

**Remark 3.1.** If there exist negative eigenvalues counted with their multiplicity $\mathcal{N}$ [8] of the operator $\Lambda^a_b = -\delta^a_b(\mathrm{d}^2/\mathrm{d}\tau^2) - R^a_{cbd}v^c v^d$ in the eigenvalue equation $-\Lambda^a_b u^b_n + \lambda_n u^a_n = 0$, the phase factor $\mathrm{e}^{\mathrm{i}\pi\sum_n(\mathrm{sign}\,\lambda_n - \mathrm{sign}\,\hat{\lambda}_n)/4}$ in the semiclassical partition function (3.21) yields the phase factor $\mathrm{e}^{-\mathrm{i}\mathcal{N}\pi/2}$.

**Proof.** Consider the eigenvalue equations of the Sturm–Liouville operator $\hat{\Lambda}^a_b = -\delta^a_b$ $(\mathrm{d}^2/\mathrm{d}\tau^2)$

$$-\hat{\Lambda}^i_j\hat{u}^j_n + \hat{\lambda}_n\hat{u}^i_n = \frac{\mathrm{d}^2\hat{u}^i_n}{\mathrm{d}\tau^2} + \hat{\lambda}_n\hat{u}^i_n = 0 \tag{3.22}$$

with

$$\hat{u}^i_n(\tau_1) = \hat{u}^i_n(\tau_2) = 0, \quad i = 1, 2, 3. \tag{3.23}$$

The solution for the differential equation (3.22) satisfying the boundary conditions (3.23) can be written as

$$\hat{u}^i_n(\tau) = c^i_n \sin\hat{\lambda}^{1/2}_n(\tau - \tau_1), \quad n = 1, 2, \ldots, \tag{3.24}$$

where the eigenvalues $\hat{\lambda}_n$ is given by

$$\hat{\lambda}_n = \left(\frac{m_n\pi}{\tau_2 - \tau_1}\right)^2, \quad m_n = 1, 2, \dots, \tag{3.25}$$

which show that sign $\hat{\lambda}_n$ is positive definite.

In the case of the positive eigenvalue $\lambda_n$ of the operator $\Lambda^a_b = -\delta^a_b(\mathrm{d}^2/\mathrm{d}\tau^2) - R^a_{cbd}v^c v^d$, there exists no phase difference since $\mathrm{sign}\,\lambda_n - \mathrm{sign}\,\hat{\lambda}_n = 0$. However, for each negative eigenvalue $\lambda_n$, we have the phase difference $\mathrm{sign}\,\lambda_n - \mathrm{sign}\,\hat{\lambda}_n = -2$ to yield the phase factor $\mathrm{e}^{-\mathrm{i}\pi/2}$. For the case of $\mathcal{N}$ negative eigenvalues $\lambda_n$, the phase factor $\mathrm{e}^{\mathrm{i}\pi\sum_n(\mathrm{sign}\,\lambda_n - \mathrm{sign}\,\hat{\lambda}_n)/4}$ yields the phase factor $\mathrm{e}^{-\mathrm{i}\mathcal{N}\pi/2}$. □

## Acknowledgements

This work is supported in part from the Korea Science and Engineering Foundation Grant R01-2001-000-00003-0.

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